Logarithmic corrections in dynamic isotropic percolation

Hans-Karl Janssen
Institut für Theoretische Physik III, Heinrich-Heine-Universität, 40225 Düsseldorf, Germany

Olaf Stenull
Department of Physics and Astronomy, University of Pennsylvania, Philadelphia, Pennsylvania 19104, USA
(Received 16 April 2003; published 26 September 2003)

Based on the field theoretic formulation of the general epidemic process, we study logarithmic corrections to scaling in dynamic isotropic percolation at the upper critical dimension $d=6$. Employing renormalization group methods we determine these corrections for some of the most interesting time dependent observables in dynamic percolation at the critical point up to and including the next to leading correction. For clusters emanating from a local seed at the origin, we calculate the number of active sites, the radius of gyration, as well as the survival probability.

DOI: 10.1103/PhysRevE.68.036131 PACS number(s): 64.60.Ak, 05.40.−a, 64.60.Ht

I. INTRODUCTION

Spreading phenomena occur in nature in many kinds with examples ranging from epidemics or forest fires [1–5] over the growth of populations [5,6] and the activity of catalyzers [7,8] to the formation of stars and galaxies [9]. In general terms, the spreading of nonconserved agents has the following scenario: An agent (e.g., an active site of a lattice, an infected individual, or a burning tree) randomly activates one or more of its neighbors. In the next time step these infected neighbors act as agents themselves and so on. Via this elementary reaction, the activation spreads out diffusively in d-dimensional space. Competition between the agents for the resources of new activations limits their local density. Moreover, an agent becomes spontaneously deactivated after some time. The long term behavior of the process, assumed to be emanating from a seed or germ at the origin, depends crucially on the difference τ between the deactivation and the activation rate. For $\tau > \tau_c$ the process spreads, approaching a homogeneous steady state, over the entire space. For $\tau < \tau_c$ the spreading will finally ebb away as an inactive extinct state is approached (with possibly a static disturbance of the initial state, but only in a finite volume). The critical point $\tau = \tau_c$ separates this endemic absorbing phase from the epidemic active phase. Spreading near $\tau = \tau_c$ constitutes a critical phenomenon and is described by universal scaling laws.

There exist two fundamental universality classes of critical spreading phenomena depending on the nature of the debris of the elementary deactivation process. If the deactivated agents can recover so that they may become newly activated (so-called simple epidemic), the process belongs to the directed percolation (DP) universality class [10] (for a review see, e.g., Ref. [11]). Here, in the active phase, one can have an epidemic surviving *in loco*. On the other hand, if the debris stays inactive forever (so-called general epidemic), the spreading process becomes locally extinct. In this case, the process belongs to the dynamic isotropic percolation (dIP) universality class and the statistics of the clusters formed by the debris are described by the usual percolation theory (for reviews see, e.g., Refs. [12,13]). Here, the epidemic cannot, of course, survive *in loco*, but an infinite epidemic is nevertheless possible in the form of a solitary wave of activity. When starting from a punctual seed, this leads to annular growth (e.g., fairy rings in two dimensions).

Most significantly, perhaps, numerical simulations and renormalization group methods have contributed to our present understanding of spreading phenomena. Until recently, accurate numerical investigations have been limited to lower spatial dimensions. However, due to the staggering pace of hardware improvements and the development of sophisticated algorithms, numerical results for spreading in high dimensions have become available [14–18]. Some of these results [17,18] clearly indicate the importance of logarithmic corrections to scaling in the upper critical dimensions $d=4$ for DP and $d=6$ for dIP. Analytic work on logarithmic corrections to percolation started not long after renormalization group methods became available. Essam *et al.* [19] determined the logarithmic corrections for the probability P_∞ of belonging to an infinite cluster, the clusters' mean-square size S and the correlation length ξ as functions of the deviation from criticality, $\tau - \tau_c$, as well as the logarithmic corrections at criticality to a "ghost" field H as a function of P_∞. Aharony [20] investigated logarithmic corrections in the context of universal amplitude ratios. More recently, Ruiz-Lorenzo [21] presented the logarithmic corrections to S and ξ at criticality as functions of the system size. In an upshot one can say that the previous results on logarithmic corrections in percolation are (i) restricted to static percolation, and (ii) limited to the leading correction.

For another system prominent in statistical physics, viz., linear polymers, logarithmic corrections observed in simulations have been very successfully described by renormalized field theory [22,23]. It turned out, however, that the leading logarithmic corrections are not sufficient to yield a satisfactory agreement with the numerical data. To the contrary, it was found that it is crucial to include the next to leading correction. We expect a similar importance of the second-order correction in the percolation problem.

The goal of this paper is to present analytical results for dIP in $d=6$ that are, with reasonable expectation, accurate enough to yield a satisfactory agreement with numerical simulations. We derive these results for some of the most interesting observables in dynamic percolation, namely, the number $N(t)$ of agents at time t generated by a seed at the spacewise and timewise origin $(\mathbf{x}=\mathbf{0},t=0)$, the survival probability $P(t)$ of the corresponding cluster, as well as the mean distance $R(t)$ of the agents from the origin (radius of gyration).

The logarithmic corrections that we study here will not be found in real physical systems because the upper critical dimension 6 for dIP does not coincide with physical dimensions, $d=2$ or $d=3$. However, our results are sure to be valuable with respect to numerical simulations. Our final analytic expressions are well suitable for comparison to numerical data. Moreover, our results define a nonuniversal time scale that signals the onset of asymptotic behavior. This time scale may be used to assess the effective significance of a given microscopic simulation model for the dIP universality class.

Complementary to the work presented here, we have investigated logarithmic corrections in DP. Our results on observables akin to the quantities studied here are certainly equally interesting and will be presented in the near future [24]. Taking a third route, we explored logarithmic corrections in static IP with emphasis on transport properties. A paper on this subject [25] will be available soon.

The outline of the present paper is the following. In Sec. II we briefly review the renormalized field theory of the general epidemic process (GEP) and previous renormalization group results on dIP. Moreover, we conduct some general considerations about logarithmic corrections in the given context. Section III hosts the core of our analysis and contains the main results. Section IV concludes the main part of this paper with a discussion of our results and several remarks. Details of our diagrammatic perturbation calculation are relegated to the Appendix.

II. RENORMALIZED FIELD THEORY OF THE GEP—A BRIEF REVIEW AND GENERAL CONSIDERATIONS ON LOGARITHMIC CORRECTIONS

In this section we briefly review the field theoretic description of the GEP and its renormalization. The aim is to provide the reader with background and to establish notation as well as known results that we need as we go along. Furthermore, we outline the general structure of the sought after logarithmic corrections.

The GEP [2,5,6] is a kinetic growth model that was introduced by Grassberger [26] as a lattice model of dIP. On a coarse grained scale, the GEP can be minimally formulated by means of the Langevin equation in the Ito sense [27]

$$\lambda^{-1}\dot{n}(\mathbf{x},t)=\nabla^2 n(\mathbf{x},t)-\tau n(\mathbf{x},t)-g n(\mathbf{x},t)\lambda \times\int_{-\infty}^{t}dt'\,n(\mathbf{x},t')+\zeta(\mathbf{x},t), \tag{2.1a}$$

$$\overline{\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t')}=\lambda^{-1}g'\,n(\mathbf{x},t)\,\delta(t-t')\,\delta(\mathbf{x}-\mathbf{x}'). \tag{2.1b}$$

Here $n(\mathbf{x},t)$ is the density of infected particles at time t and space coordinate $\mathbf{x}$. The variable τ is essentially the rate difference mentioned in the Introduction (shifted by τ_c) and specifies the deviation from criticality. λ represents a kinetic coefficient. The Gaussian random field $\zeta(\mathbf{x},t)$ subsumes reaction noise and otherwise neglected microscopic details (the overbar indicates averaging over its distribution). Its correlation respects the existence of the absorbing state. Many other analytic terms are conceivable as contributing to Eqs. (2.1). These, however, turn out to be irrelevant in the sense of the renormalization group. Especially, a diffusional noise contribution, relevant for diffusion limited reactions with multiplicative noise, can be neglected.

Langevin equations are only a convenient shorthand for a stochastic process. For the application of renormalized field theory, however, a path integral formulation of the GEP is more adequate than the Langevin equation (2.1). The dynamic functional [29–31], or in a more recent terminology the response functional, of the GEP is given [27,28] by

$$\mathcal{J}=\int d^dx\,dt\,\lambda\tilde{s}\left(\lambda^{-1}\frac{\partial}{\partial t}+(\tau-\nabla^2)+g\left(S-\frac{1}{2}\tilde{s}\right)\right)s. \tag{2.2}$$

In deriving $\mathcal{J}$ from the Langevin equation, one exploits a rescaling form invariance of the response functional $\mathcal{J}$ that allows to equate g and g'. $s(\mathbf{x},t)$ is proportional to $n(\mathbf{x},t)$. $\tilde{s}(\mathbf{x},t)$ is the response field corresponding to $s(\mathbf{x},t)$. S in the functional (2.2) stands for the density of debris, $S(\mathbf{x},t)=\lambda\int_{-\infty}^{t}dt'\,s(\mathbf{x},t')$. Note that

$$\tilde{s}(\mathbf{x},t)\leftrightarrow -S(\mathbf{x},-t) \tag{2.3}$$

is a symmetry transformation of the response functional [27].

$\mathcal{J}$ presents a vantage point for a systematic perturbation calculation in the coupling constant g. Most economically, this calculation can be done by using dimensional regularization and minimal subtraction. Using this scheme, the critical point value $\tau=\tau_c$ is formally set to zero by the perturbational expansion. Generally, τ_c is a nonanalytical function of the coupling constant g. Thus, we implicitly make the additive renormalization $\tau-\tau_c\to\tau$. For background on these methods we refer to Refs. [32,33]. An appropriate renormalization scheme is

$$s\to\mathring{s}=Z^{1/2}s,\quad \tilde{s}\to\mathring{\tilde{s}}=\tilde{Z}^{1/2}\tilde{s}, \tag{2.4a}$$

$$\lambda\to\mathring{\lambda}=Z^{-1/2}\tilde{Z}^{1/2}\lambda,\quad \tau\to\mathring{\tau}=\tilde{Z}^{-1}Z_\tau\tau, \tag{2.4b}$$

$$g\to\mathring{g}=\tilde{Z}^{-3/2}Z_u^{1/2}g,\quad G_\varepsilon g^2=u\mu^\varepsilon. \tag{2.4c}$$

Here, the $\circ$ symbol indicates un-renormalized quantities; $\varepsilon=6-d$ measures the deviation from the upper critical dimension. The factor $G_\varepsilon=\Gamma(1+\varepsilon/2)/(4\pi)^{d/2}$ is introduced exclusively for later convenience; μ is an external inverse length scale. Note that the renormalizations (2.4) preserve the invariance (2.3). The renormalization factors $\tilde{Z}$, Z_τ, and Z_u are known to three-loop order [34]. One of us [27] calculated Z to two-loop order. In the following we will need the renormalization factors explicitly to one-loop order, to which they are given by

$$\tilde{Z}=1+\frac{u}{6\varepsilon}+\cdots,\quad Z=1+\frac{4u}{3\varepsilon}+\cdots, \tag{2.5a}$$

$$Z_\tau=1+\frac{u}{\varepsilon}+\cdots,\quad Z_u=1+\frac{4u}{\varepsilon}+\cdots. \tag{2.5b}$$

The critical behavior of Green's functions $G_{n,\tilde{n}}=\langle[s]^n[\tilde{s}]^{\tilde{n}}\rangle^{(cum)}$ is governed by the Gell-Mann–Low renormalization group equation (RGE)

$$\left[\mathcal{D}_\mu+\frac{1}{2}(n\gamma+\tilde{n}\tilde{\gamma})\right]G_{n,\tilde{n}}(\{\mathbf{r},t\};\tau,u;\lambda,\mu)=0, \tag{2.6}$$

with the differential operator

$$\mathcal{D}_\mu=\mu\partial_\mu+\lambda\zeta\partial_\lambda+\tau\kappa\partial_\tau+\beta\partial_u. \tag{2.7}$$

The Wilson functions appearing in the RGE are given to two-loop order [27,34] by

$$\gamma=-\frac{4}{3}u+\left(\frac{1895}{54}+9\ln 3-5\ln 4\right)\frac{u^2}{16}, \tag{2.8a}$$

$$\tilde{\gamma}=-\frac{1}{6}u+\frac{37}{216}u^2, \tag{2.8b}$$

$$\kappa=\frac{5}{6}u-\frac{193}{108}u^2, \tag{2.8c}$$

$$\zeta=\frac{\gamma-\tilde{\gamma}}{2}=-\frac{7}{12}u+\left(\frac{1747}{54}+9\ln 3-5\ln 4\right)\frac{u^2}{32}, \tag{2.8d}$$

$$\beta=-\varepsilon u+\frac{7}{2}u^2-\frac{671}{72}u^3+\left(\frac{414\,031}{2592}+93\zeta(3)\right)\frac{3u^4}{16}. \tag{2.8e}$$

Note that we have stated β to three-loop order because the high-order contribution improves our quantitative predictions. In the remainder we will adopt a convenient abbreviated notation for the Wilson functions of the type $f(u)=f_0+f_1u+f_2u^2+\cdots$, with f standing for γ, $\tilde{\gamma}$, κ, ζ, and β, respectively. The meaning of the coefficients f_0, f_1, and so on should be evident.

The RGE can be solved by the method of characteristics. To this end one introduces a flow parameter l and sets up characteristic equations that describe how the scaling parameters transform if the external momentum scale is changed by varying $\overline{\mu}(l)=\mu l$. The characteristic for the dimensionless coupling constant u reads

$$l\frac{dw}{dl}=\beta(w), \tag{2.9}$$

where we abbreviated $w=\overline{u}(l)$. Solving this differential equation for $\varepsilon=6-d=0$ yields

$$l=l(w)=l_0w^{-\beta_3/\beta_2^2}\times\exp\left[-\frac{1}{\beta_2w}+\frac{(\beta_3^2-\beta_2\beta_4)}{\beta_2^3}w+O(w^2)\right], \tag{2.10}$$

where l_0 is an integration constant. The remaining characteristics are all of the same structure, namely,

$$l\frac{d\ln\overline{Q}(w)}{dl}=q(w). \tag{2.11}$$

Here, Q is a placeholder for Z, $\tilde{Z}$, Z_τ, and Z_λ, respectively. q is ambiguous for γ, $\tilde{\gamma}$, κ, and ζ, respectively. Exploiting $ld/dl=\beta d/dw$ we obtain the solution

$$\overline{Q}(w)=Q_0w^{q_1/\beta_2}\exp\left[\frac{(q_2\beta_2-q_1\beta_3)}{\beta_2^2}w+O(w^2)\right], \tag{2.12}$$

where Q_0 symbolizes a nonuniversal integration constant.

With the solutions to the characteristics, the scaling behavior of Green's functions is found to be

$$\begin{aligned}G_{n,\tilde{n}}(\{\mathbf{x},t\};\tau,u;\lambda,\mu)&=(\mu l)^{n(d+2)/2+\tilde{n}(d-2)/2}Z(w)^{n/2}\\&\times\tilde{Z}(w)^{\tilde{n}/2}G_{n,\tilde{n}}(\{l\mu\mathbf{x},Z_\lambda(w)\\&\times(l\mu)^2\lambda t\};Z_\tau(w)\tau/(\mu l)^2,w;1,1).\end{aligned} \tag{2.13}$$

The flow parameter introduced via the characteristics is arbitrary. Thus, we have a freedom of choice that can be exploited to rescale the relevant variables, viz., $\mathbf{x}$, t, and τ^{-1}, so that they acquire a finite asymptotic value. For the goals pursued in this paper, an appropriate choice is

$$Z_\lambda(w)(l\mu)^2\lambda t=X_0, \tag{2.14}$$

where X_0 is a constant of order unity. With this choice w and l tend to zero for $\lambda\mu^2t\to\infty$. Based on our choice (2.14) we introduce the convenient time variable

$$s=\frac{\beta_2}{2}\ln(t/t_0)=\frac{7}{4}\ln(t/t_0), \tag{2.15}$$

where $t_0\propto X_0$ is a nonuniversal time constant. From Eqs. (2.10) and (2.12), specialized to Z_λ, we get

$$s=w^{-1}-a_1\ln w+a_2w+O(w^2) \tag{2.16}$$

for the derived time variable. The constants a_1 and a_2 are given by

$$a_1=\frac{\beta_2\zeta_1-2\beta_3}{2\beta_2}=\frac{1195}{504}=2.371\,03, \tag{2.17a}$$

$$a_2=\frac{\zeta_1\beta_3-\zeta_2\beta_2}{2\beta_2}+\frac{\beta_2\beta_4-\beta_3^2}{\beta_2^2}$$

$$=\frac{1\,766\,273}{1\,016\,064}+\frac{10\ln 2-9\ln 3}{64}+\frac{279\zeta(3)}{56}=7.680\,98. \tag{2.17b}$$

Using Eq. (2.16) we obtain for the dimensionless coupling constant as a function of time the expression

$$w=s^{-1}\exp\left[a_1\frac{\ln s}{s}+O\left(\frac{\ln^2 s}{s^2},\frac{\ln s}{s^2},\frac{1}{s^2}\right)\right]. \tag{2.18}$$

Exploiting Eqs. (2.13), (2.10), (2.12), and (2.18) we find that the observables to be considered are of the form

$$\mathcal{A}=\mathcal{A}_0\exp\left[as+b\ln s+\frac{c\ln s+c'}{s}+O\left(\frac{\ln^2 s}{s^2},\frac{\ln s}{s^2},\frac{1}{s^2}\right)\right]. \tag{2.19}$$

$\mathcal{A}_0$ is, like t_0, a nonuniversal constant. a, b, c, and c' are universal numbers. a stems from mean field theory. b and c represent one- and two-loop renormalization group results, respectively. c' comprises contributions from the Wilson functions to two-loop order as well as an amplitude to be determined in an explicit one-loop calculation of $\mathcal{A}$. This amplitude depends on X_0, as do s and $\mathcal{A}_0$. Over all, a variation of X_0 leaves c' invariant.

III. LOGARITHMIC CORRECTIONS FOR THE OBSERVABLES OF INTEREST

Equipped with important intermediate results as well as some knowledge of the structure of the Green's functions $G_{n,\tilde{n}}$, we next determine the sought after logarithmic corrections. Since we already know the general form of the results, this part will be fairly brief.

A. Number of active particles

The number of active particles generated by a seed at the origin is given at criticality $\tau=0$ by

$$N(t)=\int d^dx\,G_{1,1}(\mathbf{x},t;0,u;\lambda,\mu)$$
$$=[Z(w)\tilde{Z}(w)]^{1/2}\int d^dx(\mu l)^d$$
$$\times G_{1,1}(l\mu\mathbf{r},Z_\lambda(w)(l\mu)^2\lambda t;0,w;1,1)$$
$$=[Z(w)\tilde{Z}(w)]^{1/2}G_{1,1}(\mathbf{q}=0,X_0;0,w;1,1). \tag{3.1}$$

Specializing solution (2.12) to $Q=Z$ and $Q=\tilde{Z}$, we obtain

$$[Z(w)\tilde{Z}(w)]^{1/2}\propto\exp\left[\frac{(\gamma_1+\tilde{\gamma}_1)}{2\beta_2}\ln w+\left(\frac{(\gamma_2+\tilde{\gamma}_2)\beta_2-(\gamma_1+\tilde{\gamma}_1)\beta_3}{2\beta_2^2}\right)w+O(w^2)\right]. \tag{3.2}$$

A perturbation expansion of the Green's function $G_{1,1}$ brings about an amplitude $A_N(X_0)$ that we define via

$$G_{1,1}(\mathbf{q}=0,X_0;0,w;1,1)\propto[1+A_N(X_0)w+O(w^2)]. \tag{3.3}$$

This amplitude follows from our one-loop calculation presented in Appendix as

$$A_N(X_0)=\frac{3}{8}\left(\mathcal{Z}+\frac{3}{2}-\frac{\ln 2}{3}\right), \tag{3.4}$$

where we used the shorthand notation $\mathcal{Z}=\ln X_0+C_E$, with C_E being Euler's constant. Collecting Eqs. (3.1)–(3.3), we find

$$N(t)=N_0(w^{-1}+B_N)^{a_N}\exp[c_Nw+O(w^2)]$$
$$=N_0'(s+B_N)^{a_N}\left[1+\frac{b_N\ln s+c_N}{s}+O\left(\frac{\ln^2 s}{s^2},\frac{\ln s}{s^2},\frac{1}{s^2}\right)\right], \tag{3.5}$$

where N_0 is a nonuniversal constant, N_0' is a nonuniversal constant slightly different from N_0, and $B_N=A_N/a_N$. The first row of Eq. (3.5) and the result (2.16) constitute a parametric representation of the tuple (N,s) that is suitable for comparison to numerical simulations. The second row of Eq. (3.5) shows the more traditional form. The constants a_N, b_N, c_N, and B_N are given by

$$a_N=-\frac{\gamma_1+\tilde{\gamma}_1}{2\beta_2}=\frac{3}{14}=0.214\,286, \tag{3.6a}$$

$$b_N=a_N\frac{2\beta_3-\zeta_1\beta_2}{2\beta_2}=-\frac{1195}{2352}=-0.508\,078, \tag{3.6b}$$

$$c_N=\frac{\gamma_2+\tilde{\gamma}_2}{2\beta_2}-\beta_3\frac{\gamma_1+\tilde{\gamma}_1}{2\beta_2^2}$$
$$=-\frac{365}{1568}+\frac{9\ln 3-5\ln 4}{112}$$
$$=-0.206\,387, \tag{3.6c}$$

$$B_N=\frac{7}{4}\left(\mathcal{Z}+\frac{3}{2}-\frac{\ln 2}{3}\right)=1.75\mathcal{Z}+2.220\,66. \tag{3.6d}$$

Note from Eqs. (2.15) and (3.5) that the arbitrary constant $\mathcal{Z}$ could be eliminated by a rescaling of the nonuniversal time constant t_0. This finding will also apply to the remaining results stated below.

At this point we would like to warn against attempts to deduce the logarithmic corrections for dynamic quantities from the logarithmic corrections calculated for static percolation. Grassberger [15], for example, exploited the results of Essam *et al.* [19] via replacing τ by $1/t$ on the grounds that the critical exponent ν_t for the correlation time is one in mean-field theory. This reasoning leads to $N(t)\sim[\ln(t)]^{2/7}$ [35]. From Eqs. (3.5) and (3.6a), however, we see that the correct result, to leading order, is $N(t)\sim[\ln(t/t_0)]^{3/14}$. By merely using the mean-field relation between τ and t one misses contributions to the leading logarithmic term stemming from renormalization factors including Z [cf. Eq. (3.2)]. Since Z is absent in static percolation [36], one cannot deduce the logarithmic behavior of the dynamic quantity $N(t)$ from the known results for static percolation. Likewise, it should not be attempted to combine our dynamic results, e.g., those for $N(t)$ and $R(t)$, to obtain predictions on logarithmic corrections in static percolation.

B. Radius of gyration

The mean square distance from the origin of the active particles is defined as

$$R(t)^2=\frac{\int d^dx\,\mathbf{x}^2G_{1,1}(\mathbf{x},t)}{2d\int d^dx G_{1,1}(\mathbf{x},t)}=-\left.\frac{\partial\ln G_{1,1}(\mathbf{q},t)}{\partial q^2}\right|_{\mathbf{q}=\mathbf{0}}. \tag{3.7}$$

From the scaling form (2.13), it follows for $\tau=0$ that

$$\begin{aligned}\left.\frac{\partial\ln G_{1,1}(\mathbf{q},t)}{\partial q^2}\right|_{\mathbf{q}=\mathbf{0}}&=\left.\frac{\partial\ln G_{1,1}((l\mu)^{-1}\mathbf{q},Z_\lambda(w)(l\mu)^2\lambda t;0,w;1,1)}{\partial q^2}\right|_{\mathbf{q}=\mathbf{0}}\\&=(l\mu)^{-2}\left.\frac{\partial\ln G_{1,1}(\mathbf{q},X_0;0,w;1,1)}{\partial q^2}\right|_{\mathbf{q}=\mathbf{0}}.\end{aligned} \tag{3.8}$$

Incorporating the solutions to the appropriate characteristics and the results of the Appendix,

$$\begin{aligned}-\frac{\partial}{\partial q^2}\ln G_{1,1}(\mathbf{q},\lambda\mu^2t=X_0;0,w;1,1)\big|_{\mathbf{q}=\mathbf{0}}\\=X_0[1+A_R(X_0)w+O(w^2)],\end{aligned} \tag{3.9}$$

with

$$A_R(X_0)=\frac{7}{24}\left(\mathcal{Z}-\frac{2}{3}-\frac{\ln 2}{7}\right), \tag{3.10}$$

we find

$$\begin{aligned}t^{-1}R^2&=R_0^2(w^{-1}+B_R)^{a_R}\exp(c_Rw+O(w^2))\\&=R_0'^2(s+B_R)^{a_R}\left[1+\frac{b_R\ln s+c_R}{s}\right.\\&\quad\left.+O\left(\frac{\ln^2 s}{s^2},\frac{\ln s}{s^2},\frac{1}{s^2}\right)\right],\end{aligned} \tag{3.11}$$

with R_0^2 and $R_0'^2$ being nonuniversal amplitudes. Here the constants a_R, b_R, c_R, and $B_R=A_R/a_R$ are given by

$$a_R=-\frac{\zeta_1}{\beta_2}=\frac{1}{6}=0.166\,666, \tag{3.12a}$$

$$b_R=a_R\frac{2\beta_3-\zeta_1\beta_2}{2\beta_2}=-\frac{1195}{3024}=-0.395\,172, \tag{3.12b}$$

$$c_R=\frac{\zeta_2}{\beta_2}-\frac{\zeta_1\beta_3}{\beta_2^2}=-\frac{937}{6048}+\frac{9\ln 3-5\ln 4}{112}=-0.128\,534, \tag{3.12c}$$

$$B_R=\frac{7}{4}\left(\mathcal{Z}-\frac{2}{3}-\frac{\ln 2}{7}\right)=1.75\mathcal{Z}-1.339\,95. \tag{3.12d}$$

C. Survival probability

As shown in Ref. [37], the survival probability of an active cluster emanating from a seed at the origin is given by

$$P(t)=-\lim_{k\to\infty}\langle e^{-k\mathcal{N}}\tilde{s}(-t)\rangle, \tag{3.13}$$

where $\mathcal{N}=\int d^dxs(\mathbf{x},0)$. For the purpose of actual calculations, it is convenient to rewrite Eq. (3.13) as

$$P(t)=-\lim_{k\to\infty}\langle\tilde{s}(-t)\rangle_k=-G_{0,1}(-t,\tau,k=\infty,u;\lambda,\mu), \tag{3.14}$$

where $\langle\cdots\rangle_k$ stands for averaging with respect to the response functional $\mathcal{J}_k$ that is obtained upon augmenting the original response functional (2.2) by a source $k(t)=k\delta(t)$ conjugate to the field s:

$$\mathcal{J}_k=\mathcal{J}+\int dt k(t)\mathcal{N}(t). \tag{3.15}$$

With this source present, one no longer has $\langle\tilde{s}\rangle=0$. To avoid tadpoles in our perturbation calculation, we perform a shift $\tilde{s}\to\tilde{s}+\tilde{M}$ so that $\langle\tilde{s}\rangle=0$ is restored. This procedure leads to the new response functional

$$\mathcal{J}_k = \int d^dx dt \bigg[\lambda \tilde{s} \bigg(\lambda^{-1} \frac{\partial}{\partial t} + (\tau - g\tilde{M} - \nabla^2) + g\bigg(S - \frac{1}{2}\tilde{s} \bigg) \bigg) s + \lambda g \tilde{M} s S + \bigg(-\dot{\tilde{M}} + \lambda \tau \tilde{M} - \frac{\lambda g}{2}\tilde{M}^2 + k \bigg) s \bigg]. \quad (3.16)$$

Based on this functional we calculate $G_{0,1} = \tilde{M}$ to one-loop order. Some details of this calculation are in the Appendix. We obtain

$$G_{0,1}(-X_0,0,k=\infty,w;1,1) \propto w^{-1/2}[1 + A_P(X_0)w + O(w^2)], \quad (3.17)$$

with the amplitude $A_P(X_0)$ reading

$$A_P(X_0) = \frac{5}{8}\bigg(\mathcal{Z} + 1 - \frac{11 \ln 2}{5} \bigg). \quad (3.18)$$

Recalling the scaling form (2.13) and our choice for the flow parameter we deduce that, for $\tau = 0$,

$$P(t) = -\tilde{Z}(w)^{1/2}(\mu l)^2 G_{0,1}(-X_0,0,\infty,w;1,1). \quad (3.19)$$

Collecting, we then obtain

$$tP(t) = P_0(w^{-1} + B_P)^{a_P} \exp[c_P w + O(w^2)] = P_0'(s + B_P)^{a_P}\bigg[1 + \frac{b_P \ln s + c_P}{s} + O\bigg(\frac{\ln^2 s}{s^2}, \frac{\ln s}{s^2}, \frac{1}{s^2} \bigg) \bigg]. \quad (3.20)$$

P_0 and P_0' are simply related nonuniversal amplitudes. The constants a_P, b_P, c_P, and $B_P = A_P / a_P$ are given by

$$a_P = \frac{2\zeta_1 + \beta_2 - \tilde{\gamma}_1}{2\beta_2} = \frac{5}{14} = 0.357\,143, \quad (3.21a)$$

$$b_P = a_P \frac{2\beta_3 - \zeta_1\beta_2}{2\beta_2} = -\frac{5975}{7056} = -0.846\,797, \quad (3.21b)$$

$$c_P = \frac{\tilde{\gamma}_2 - 2\zeta_2}{2\beta_2} + \beta_3 \frac{2\zeta_1 - \tilde{\gamma}_1}{2\beta_2^2} = \frac{1637}{14\,112} - \frac{9\ln 3 - 5\ln 4}{112} = 0.089\,607, \quad (3.21c)$$

$$B_P = \frac{7}{4}\bigg(\mathcal{Z} + 1 - \frac{11\ln 2}{5} \bigg) = 1.75\mathcal{Z} - 0.918\,617. \quad (3.21d)$$

IV. DISCUSSION OF RESULTS AND CONCLUDING REMARKS

As far as time dependent observables in percolation are concerned, we are not aware of any previous analytic work addressing logarithmic corrections. More general, we do not know of any work that has determined logarithmic corrections in percolation (static or dynamic, IP or DP) beyond the leading corrections. Here we went beyond the leading terms, and hence we are confident that our results compare well with simulations, perhaps even quantitatively. For linear polymers it turned out that the knowledge of the leading logarithmic correction is not sufficient for a good agreement between simulation data and theory. Rather, the next to leading corrections turned out to be crucial in comparing numerical and analytical results. We expect the same for percolation. Indeed, preliminary Monte Carlo results corroborate this expectation [17,18].

Our results define a nonuniversal time scale t_0. For times t greater than t_0, we expect the validity of our asymptotic expansions. The time scale t_0 can be utilized as a measure of quality for different microscopic models of dynamical percolation. Thus, our results may guide those performing simulations in choosing the most efficient model.

It is interesting to note that the time scale t_0 has an analog in quantum chromodynamics. For times greater than t_0, the model becomes asymptotically free. Thus, with the exchange of an infrared-free theory to an ultraviolet-free theory, t_0 corresponds to the hadronization scale of quantum chromodynamics. The dependence of our results on this dimensional nonuniversal parameter t_0 parallels therefore the phenomenon of dimensional transmutation in renormalizable asymptotically free quantum field theories that are naively scale-free.

Our results feature a mutual nonuniversal constant, viz., $\mathcal{Z}$. This constant could be eliminated by rescaling t_0. One might be tempted to think that one could eliminate the entire amplitudes $A_N(X_0)$, and so on, from our results by rescaling t_0, and that the calculation of the amplitudes is hence superfluous. One has to keep in mind, however, that one has to choose t_0 consistently for all observables. Thus, one cannot remove the amplitudes simultaneously from all the observables, and their calculation is indeed necessary.

We refrain from eliminating $\mathcal{Z}$ from our results because it might be exploited, due to its nonuniversality, as a fit parameter. By fitting $\mathcal{Z}$ one can compensate partially for the effect of higher order terms that have been neglected in our calculations. In this sense one can think of $\mathcal{Z}$ as mimicking these higher order terms.

When written as an explicit function of time, the observables of interest have fairly complicated formulas. Using the parametric representation in terms of the effective coupling constant w eases this situation. Moreover, the time and the observables possess a nicely systematic expansion in w so that it is straightforward to keep track of the different orders in perturbation theory. In the traditional from, the orders are not so clear cut because nested functions of logarithms have to be compared. The parametric representations can be conveniently compared to simulations. Essentially, one just needs to make parametric plots of the tuples (N,s), (R,s), and (P,s), and then compare the numerical data to these plots.

In order to improve the accuracy of our results, one needs a refined quantitative knowledge of the Wilson functions. Whereas the ε expansion results for the critical percolation

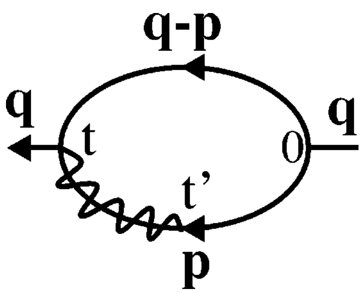

FIG. 1. Self-energy $\Sigma(\mathbf{q},t)$ at one-loop order.

exponents have been improved by resummation techniques such as Padé-Borel resummation [34], this kind of refinement has not yet been achieved for the percolation Wilson functions. Here lies an opportunity for useful future work. Another possibility for future work is to improve the results on static percolation mentioned in the Introduction by calculating the next to leading logarithmic corrections. With the kind of field theoretic methods that we applied here, this is a reasonable task.

Apparently, firm numerical results that are suitable for comparison to our analytical results are not available yet. We hope, however, that our work triggers increasing efforts in this direction, and that corresponding numerical results will be available in the near future.

ACKNOWLEDGMENTS

This work was supported by the Deutsche Forschungsgemeinschaft via the Sonderforschungsbereich 237 "Unordnung und große Fluktuationen" and the Emmy Noether Programm. We thank D. Stauffer for bringing our attention to several references.

APPENDIX: EXPLICIT CALCULATION OF GREEN'S FUNCTIONS

In this appendix we outline our one-loop calculation of the scaling functions belonging to the Green's functions $G_{1,1}$ and $G_{0,1}$. In particular, we compute the amplitudes $A_N(X_0)$, $A_P(X_0)$, and $A_R(X_0)$ entering the logarithmic corrections.

1. Green's function $G_{1,1}$

A first step of any diagrammatic perturbation calculation is, of course, the determination of the constituting elements. From the response functional (2.2) we gather the Gaussian propagator

$$G(\mathbf{q},t)=\theta(t)\exp[-\lambda(\tau+q^2)t] \tag{A1}$$

and the three-leg vertices λg and $-\lambda^2 g\,\theta(t-t')$, where $\theta(t)$ denotes the step function. With these elements, the self-energy $\Sigma(\mathbf{q},t)$ is given at one-loop order by the diagram depicted in Fig. 1. This diagram stands for the mathematical formula

$$\Sigma(\mathbf{q},t)=-\lambda^3 g^2\int_0^t dt'\int_{\mathbf{p}} G(\mathbf{p},t')G(\mathbf{q-p},t). \tag{A2}$$

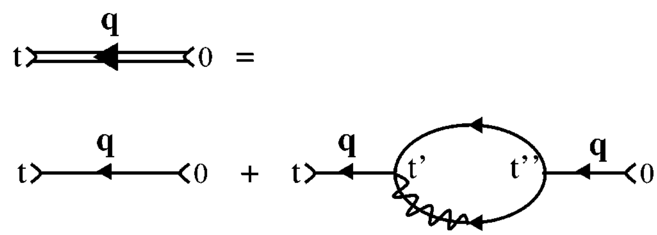

FIG. 2. Dyson equation (A4) to one-loop order.

After integrating out the loop momentum we can rewrite $\Sigma(\mathbf{q},t)$ as

$$\Sigma(\mathbf{q},t)=-\frac{(\lambda g)^2}{(4\pi)^{d/2}}(\lambda t)^{1-d/2}\times\int_0^1 ds\frac{\exp\{-\lambda t[(1+s)\tau+sq^2/(1+s)]\}}{(1+s)^{d/2}}. \tag{A3}$$

For our purposes we need Green's or connected correlation functions rather than vertex functions. Hence, we have to consider Feynman diagrams with external legs attached rather than amputated diagrams. The Green's function $G_{1,1}$ is determined by the Dyson equation

$$G_{1,1}(\mathbf{q},t)=G(\mathbf{q},t)+\int_0^t dt'\int_0^{t'} dt'' G(\mathbf{q},t-t')\times\Sigma(\mathbf{q},t'-t'')G(\mathbf{q},t'')+\cdots. \tag{A4}$$

A diagrammatic representation of the Dyson equation is given in Fig. 2. Upon substituting Eq. (A3) into (A4) we obtain after an integration

$$G_{1,1}(\mathbf{q},t)=G(\mathbf{q},t)\left[1-\frac{u(\lambda\mu^2 t)^{\varepsilon/2}}{\Gamma(1+\varepsilon/2)}\int_0^1\frac{ds}{(1+s)^{d/2}}\times\int_0^1 dx(1-x)x^{1-d/2}\exp[\alpha(s)x]\right]. \tag{A5}$$

Here, we used the shorthand notation

$$\alpha(s)=\left(\frac{q^2}{1+s}-(1+s)\tau\right)\lambda t. \tag{A6}$$

Now, we set $\tau=0$ and expand Eq. (A5) to order q^2. The integrations are easily performed. After ε expansion we get

$$G_{1,1}(\mathbf{q},t)=G(\mathbf{q},t)\left\{1+\frac{u(\lambda\mu^2 t)^{\varepsilon/2}}{\Gamma(1+\varepsilon/2)}\left[\left(\frac{3}{4\varepsilon}+\frac{9}{16}-\frac{\ln 2}{8}\right)-\left(\frac{7}{12\varepsilon}-\frac{7}{36}-\frac{\ln 2}{24}\right)\lambda q^2 t\right]\right\}. \tag{A7}$$

The next step is to remove the ε poles by employing the renormalization scheme (2.4). Letting $G_{1,1}\to\mathring{G}_{1,1}$, $\lambda\to\mathring{\lambda}$, and using

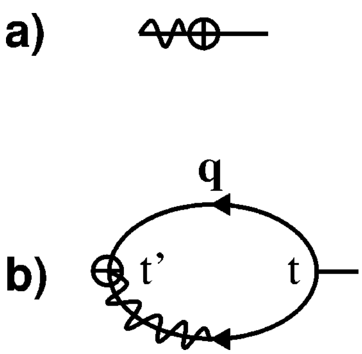

FIG. 3. (a) The new vertex $-\lambda^2 g\tilde{M}\theta(t-t')$ and (b) the one-loop tadpole diagram T(t).

$$G_{1,1}=(\tilde{Z}Z)^{-1/2}\mathring{G}_{1,1}=\left(1-\frac{3u}{4\varepsilon}\right)\mathring{G}_{1,1} \tag{A8}$$

as well as

$$\mathring{\lambda}=(\tilde{Z}/Z)^{1/2}\lambda=\left(1-\frac{7u}{12\varepsilon}\right)\lambda, \tag{A9}$$

we observe that the ε poles are indeed removed. For the renormalized Green's function we obtain

$$G_{1,1}(\mathbf{q},t)=G(\mathbf{q},t)\left\{1+\frac{3u}{8}\left(\ln(\lambda\mu^2 t)+C_E+\frac{3}{2}-\frac{\ln 2}{3}\right)-\frac{7u}{24}\left(\ln(\lambda\mu^2 t)+C_E-\frac{2}{3}-\frac{\ln 2}{7}\right)\lambda q^2 t\right\}. \tag{A10}$$

Two results important for the logarithmic correction can be extracted from Eq. (A10). Upon setting $\mathbf{q}=0$ we find

$$G_{1,1}(\mathbf{q}=0,\lambda\mu^2 t=X_0;\tau=0,w;1,1)=1+\frac{3}{8}\left(\mathcal{Z}+\frac{3}{2}-\frac{\ln 2}{3}\right)w, \tag{A11}$$

and hence the amplitude $A_N(X_0)$ as stated in Eq. (3.3). Moreover, we get

$$-X_0^{-1}\frac{\partial}{\partial q^2}\ln G_{1,1}(\mathbf{q},\lambda\mu^2 t=X_0;\tau=0,w;1,1)\big|_{q^2=0}=1+\frac{7}{24}\left(\mathcal{Z}-\frac{2}{3}-\frac{\ln 2}{7}\right)w, \tag{A12}$$

which leads to our result for $A_R(X_0)$ given in Eq. (3.10).

2. Green's function $G_{0,1}$

Now we determine $G_{0,1}$ as required in Eq. (3.14). The diagrammatic elements associated with the functional (3.16) comprise the two vertices encountered above. In addition, there is a third vertex, viz., the one depicted in Fig. 3(a). The Gaussian propagator for the new functional has to be determined from the differential equation

$$[\lambda^{-1}\partial_t+\tau-g\tilde{M}(t)+q^2]\bar{G}(\mathbf{q},t,t')=\lambda^{-1}\delta(t-t'). \tag{A13}$$

To avoid tadpoles, $\tilde{M}(t)$ has to satisfy the differential equation

$$\dot{\tilde{M}}(t)-\lambda\tau\tilde{M}(t)+\frac{\lambda g}{2}\tilde{M}(t)^2-k(t)+T(t)=0. \tag{A14}$$

At one-loop order, the tadpole $T(t)$ is given by the diagram shown in Fig. 3(b).

The initial and terminal conditions for the fields necessitate the ansatz $\tilde{M}(t)=-\theta(-t)K(-t)^{-1}$. The type of the source term, $k(t)=k\delta(t)$ with $k\to\infty$, demands the initial condition $K(0)=0$. With this information, the differential equation (A14) can be transformed without much effort into the integral equation

$$K(t)+\frac{g}{2\tau}=e^{\lambda\tau t}\left(\int_0^t dt'\,e^{-\lambda\tau t'}K(t')^2T(-t')+\frac{g}{2\tau}\right). \tag{A15}$$

At mean field level, the solution to Eq. (A15) is given by

$$K_0(t)=\frac{g}{2\tau}(e^{\lambda\tau t}-1). \tag{A16}$$

Inserting the corresponding $\tilde{M}_0(t)=-K_0(-t)^{-1}$ into the differential equation (A13), we find the modified Gaussian propagator

$$\bar{G}_0(\mathbf{q},t,t')=\theta(t-t')\left(\frac{K_0(-t)}{K_0(-t')}\right)^2\exp[\lambda(\tau-q^2)(t-t')]. \tag{A17}$$

Having the modified Gaussian propagator at our demand, we are now in the position to calculate the diagram depicted in Fig. 3(b). Eventually, we obtain

$$K(t)^2T(-t)=\lambda^3g^2K_0(t)^{-2}\int_0^t dt'\int_{t'}^t dt''\times\frac{K_0(t')K_0(t'')^2\exp[\lambda\tau(2t-t'-t'')]}{[4\pi\lambda(2t-t'-t'')]^{d/2}}. \tag{A18}$$

The further evaluation of Eq. (A18) is fairly straightforward for $\tau=0$. Away from criticality, the calculation is more challenging and will be addressed in a future publication [38]. Here, we find for $\tau=0$ and after ε expansion

$$K(t)^2T(-t)=-\frac{\lambda g^3(\lambda t)^{\varepsilon/2}}{(4\pi)^{d/2}}\left(\frac{5}{8\varepsilon}+\frac{5}{8}-\frac{11\ln 2}{16}\right). \tag{A19}$$

Insertion of this intermediate result into Eq. (A15) yields

$$K(t)=\frac{g\lambda t}{2}\left[1-\frac{u(\lambda\mu^2t)^{\varepsilon/2}}{\Gamma(1+\varepsilon/2)}\left(\frac{5}{4\varepsilon}+\frac{5}{8}-\frac{11\ln 2}{8}\right)\right]. \tag{A20}$$

Next, we renormalize. Indicating the consistency of our previous steps, the appropriate combination of renormalization factors $(Z\tilde{Z}/Z_u)^{-1/2}=1+5u/(4\varepsilon)+\cdots$ cancels the ε pole in Eq. (A20). The renormalized $K(t)$ reads

$$K(t)=\frac{g\lambda t}{2}\left[1-\frac{5u}{8}\left(\ln(\lambda\mu^2t)+C_E+1-\frac{11\ln 2}{5}\right)\right]. \tag{A21}$$

Exploiting $G_{0,1}(-t)=K(t)^{-1}$ and $\lambda\mu^2t=X_0$ as well as recalling the definition of $\mathcal{Z}$, we finally obtain

$$\frac{(4\pi)^{3/2}X_0}{2}G_{0,1}(-\lambda\mu^2t=X_0;\tau=0,w;1,1)=w^{-1/2}\left[1+\frac{5}{8}\left(\mathcal{Z}+1-\frac{11\ln 2}{5}\right)w\right]. \tag{A22}$$

[1] S.R. Broadbent and J.M. Hammersley, Proc. Cambridge Philos. Soc. **53**, 629 (1957).
[2] D. Mollison, J. R. Stat. Soc. Ser. B. Methodol. **39**, 283 (1977).
[3] T. M. Liggett, *Interacting Particle Systems* (Springer, New York, 1985).
[4] R. Durrett, *Lectures on Particle Systems and Percolation* (Wadsworth, Pacific Grove, CA, 1988).
[5] N. T. J. Bailey, *The Mathematical Theory of Infectious Diseases* (Griffin, London, 1985).
[6] J. D. Murray, *Mathematical Biology* (Springer, Berlin, 1989).
[7] R.M. Ziff, E. Gulari, and Y. Barshad, Phys. Rev. Lett. **56**, 2553 (1986).
[8] I. Jensen, H.C. Fogedby, and R. Dickman, Phys. Rev. A **41**, 3411 (1990).
[9] L.S. Schulman and P.E. Seiden, J. Stat. Phys. **19**, 293 (1978).
[10] H.K. Janssen, Z. Phys. B: Condens. Matter **42**, 151 (1981).
[11] H. Hinrichsen, Adv. Phys. **49**, 815 (2001).
[12] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor & Francis, London, 1994).
[13] *Fractals and Disordered Systems*, edited by A. Bunde and S. Havlin (Springer, Berlin, 1996).
[14] G. Paul, R.M. Ziff, and H.E. Stanley, Phys. Rev. E **64**, 026115 (2001).
[15] P. Grassberger, Phys. Rev. E **67**, 036101 (2003).
[16] D. Osterkamp, D. Stauffer, and A. Aharony, e-print cond-mat/0302051.
[17] P. Grassberger (private communication).
[18] S. Lübeck (private communication).
[19] J.W. Essam, D.S. Gaunt, and A.J. Guttmann, J. Phys. A **11**, 1983 (1978).
[20] A. Aharony, Phys. Rev. B **22**, 400 (1980).
[21] J.J. Ruiz-Lorenzo, J. Phys. A **31**, 8773 (1998).
[22] P. Grassberger, R. Hegger, and L. Schäfer, J. Phys. A **27**, 7265 (1994).
[23] J. Hager and L. Schäfer, Phys. Rev. E **60**, 2071 (1999).
[24] H. K. Janssen and O. Stenull (unpublished).
[25] O. Stenull and H. K. Janssen, Phys. Rev. E (to be published).
[26] P. Grassberger, Math. Biosci. **62**, 157 (1983).
[27] H.K. Janssen, Z. Phys. B: Condens. Matter **58**, 311 (1985).
[28] J.L. Cardy and P. Grassberger, Z. Phys. A **18**, L267 (1985).
[29] H.K. Janssen, Z. Phys. B: Condens. Matter **23**, 377 (1976); R. Bausch, H.K. Janssen, and H. Wagner, *ibid.* **24**, 113 (1976).
[30] C. DeDominicis, J. Phys. (Paris), Colloq. **37**, C-247 (1976); C. DeDominicis and L. Peliti, Phys. Rev. B **18**, 353 (1978).
[31] H. K. Janssen, in *Dynamical Critical Phenomena and Related Topics*, edited by C. P. Enz, Lecture Notes in Physics Vol. 104 (Springer, Heidelberg, 1979); H. K. Janssen, in *From Phase Transition to Chaos*, edited by G. Györgyi, I. Kondor, L. Sasvári, and T. Tél (World Scientific, Singapore, 1992).
[32] D. J. Amit, *Field Theory, the Renormalization Group, and Critical Phenomena* (World Scientific, Singapore, 1984).
[33] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon, Oxford, 1996).
[34] O.F. de Alcantara Bonfim, J.E. Kirkham, and A.J. McKane, J. Phys. A **13**, L247 (1980); **14**, 2391 (1981).
[35] In Ref. [15] the number of active sites is denoted by *M*.
[36] The field theory of static percolation is founded on the Potts model. The role of the order parameter field is played by a Potts-spin-like field, say, ϕ. With the notation we use here, ϕ is renormalized by the renormalization factor $\tilde{Z}$, viz., $\phi\rightarrow\mathring{\phi}=\tilde{Z}^{1/2}\phi$.
[37] H. K. Janssen (unpublished).
[38] H. K. Janssen and O. Stenull (unpublished).